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Cumulative Variable Formulation for Transient Conductive and Radiative Transport in Participating Media

Jay I. Frankel*
University of Tennessee, Knoxville, Tennessee 37996-2210

A new mathematical formulation is proposed for transient conductive and radiative transport in a participating gray, isotropically scattering plane-parallel medium. The methodology can be easily extended to include numerous additional effects. A systematic and unified treatment is presented using cumulative variables that allows for high-order integration using standard initial-value methods in the temporal variable while allowing for an effective orthogonal collocation method to be implemented in the spatial variable. A spectral approach is incorporated in the present context where Chebyshev polynomials of the first kind are used as the basis functions. This article illustrates the methodology and presents some comparisons with previously reported works.

Nomenclature

A = coefficient matrix, Eq. (20b)
$A_m^\alpha(\eta)$ = function defined in Eq. (14d)
a_{jm} = matrix entries for A, Eq. (20c)
$a_m(\xi)$ = exact expansion coefficients, Eq. (11a)
$a_m^N(\xi)$ = approximate expansion coefficients, Eq. (13a)
$\bar{a}(\xi)$ = vector containing $a_m^N(\xi)$, $m = 0, 1, \ldots, N$
B = coefficient matrix, Eq. (22a)
b_{jm} = matrix entries for B, Eq. (22b)
$b_m(\xi)$ = exact expansion coefficients, Eq. (11b)
$b_m^N(\xi)$ = approximate expansion coefficients, Eq. (13b)
$\bar{b}(\xi)$ = vector containing $b_m^N(\xi)$, $m = 0, 1, \ldots, N$
$c_m(\xi)$ = exact expansion coefficients, Eq. (11c)
$c_m^N(\xi)$ = approximate expansion coefficients, Eq. (13c)
c_p = specific heat
$\bar{c}(\xi)$ = vector containing $c_m^N(\xi)$, $m = 0, 1, \ldots, N$
$E_n(z)$ = nth exponential integral function
$f_j^N(\xi)$ = entries for vector $\bar{f}(\xi)$, Eq. (20d)
$\bar{f}(\xi)$ = vector containing $f_j^N(\xi)$, $j = 0, 1, \ldots, N$
$G^*(\tau, \xi)$ = $[\tilde{G}(\tau, \xi)/4n^2\sigma T_r^4]$, dimensionless incident radiation function, Ref. 11
$\tilde{G}(\tau, \xi)$ = incident radiation function defined in Ref. 11
$G(\eta, \xi)$ = $G^*[\alpha(1 + \eta), \xi]$, dimensionless Chebyshev incident radiation function
$g_j^N(\xi)$ = entries for vector $\bar{g}(\xi)$, Eqs. (22c–22e)
$\bar{g}(\xi)$ = vector containing $g_j^N(\xi)$, $j = 0, 1, \ldots, N$
$h_j^N(\xi)$ = entries for vector $\bar{h}(\xi)$, Eq. (22f)
$\bar{h}(\xi)$ = vector containing $h_j^N(\xi)$, $j = 0, 1, \ldots, N$
$h(\eta, \xi)$ = function defined by Eq. (5c)
k = thermal conductivity
l = dimensional length of plate
N = Nth-order approximation
N_{cr} = $(k\beta/4n^2\sigma T_r^3)$, conduction-radiation number, Ref. 11
n = index of refraction
$p_j^N(\xi)$ = entries for vector $\bar{p}(\xi)$, Eq. (26d)
$\bar{p}(\xi)$ = vector containing $p_j^N(\xi)$, $j = 0, 1, \ldots, N$
$Q^r(\eta, \xi)$ = $\tilde{Q}^r[\alpha(1 + \eta), \xi]$, dimensionless radiative heat flux in Chebyshev domain
$Q_N^r(\eta, \xi)$ = Nth-order approximation to $Q^r(\eta, \xi)$
$\tilde{Q}^r(\tau, \xi)$ = $(q_r''/4n^2\sigma T_r^4)$, dimensionless radiative heat flux, Ref. 11
q_r'' = dimensional radiative heat flux
$R_k^N(\eta, \xi)$ = local residual function, $k = 1, 2, 3$
s = dummy variable
$T_m(\eta)$ = mth Chebyshev polynomial of the first kind
T_r = reference temperature
t = time
$W_{1,m}^\alpha(\eta)$ = function defined by Eq. (27b)
$W_{2,m}^\alpha(\eta)$ = function defined by Eq. (27c)
y = dimensional spatial coordinate
α = dimensionless half depth, $\tau_D/2$
β = extinction coefficient, Ref. 11
$\bar{\gamma}(\xi)$ = vector containing $\gamma_j^N(\xi)$, $j = 0, 1, \ldots, N$
$\gamma_j^N(\xi)$ = approximate expansion coefficients, Eq. (25)
δ = Dirac delta function
η = $(\tau/\alpha) - 1$, dimensionless (Chebyshev domain) spatial variable
η_j = jth collocation point defined by Eq. (17b)
η_0 = dummy variable
θ_i = initial temperature at $\xi = 0$
$\theta_N(\eta, \xi)$ = Nth-order approximation to $\theta(\eta, \xi)$
$\theta(\eta, \xi)$ = $\tilde{\theta}[\alpha(1 + \eta), \xi]$, dimensionless temperature in Chebyshev domain
$\tilde{\theta}(\tau, \xi)$ = (T/T_r), dimensionless temperature, Ref. 11
θ_2 = imposed boundary condition at $\eta = 1$
ξ = $(k/\rho c_p)\beta^2 t$, dimensionless time, Ref. 11
ξ_0 = dummy variable
ρ = density
σ = Stefan-Boltzmann constant
τ = βy, optical variable, Ref. 11
τ_D = βl, optical depth, Ref. 11
$\Psi_k(\eta, \xi)$ = cumulative variables, $k = 1, 2, 3$
ω = single-scattering albedo

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*Associate Professor, Mechanical and Aerospace Engineering Department. Member AIAA.

I. Introduction

COMPUTATIONAL investigations studying transient combined radiative and conductive transport in participating media have recaptured the interest and attention of researchers over the past 10 years. This rejuvenation is in part due to new and exciting engineering applications requiring detailed knowledge of temperature and flux distributions inside a medium. The computational prowess of today's machines allow researchers to perform detailed analyses and obtain clear graphical outputs from which interpretation is quickly realized. Today, scientific endeavors requiring dynamical considerations include the study of: heat transfer in ceramic diesel liners,[1] transient responses to volumetrically scattering heat shields,[2,3] transient studies of high-temperature windows,[4] de-icing of solids through radiant heating,[5] transient combustion of fuel droplets,[6] dynamic investigations involving packed-beds, transient responses in active thermal insulation systems,[7] and porous thermal insulations,[8] as well as numerous other physical applications. Additionally, thermomechanical aspects involving semitransparent materials have been investigated for practical assessment in numerous applications.[9]

From a review of the literature, the typical setting for most computational investigations begins with the conventional differential form of the heat equation and either the integral or integro-differential form of the linearized Boltzmann transport equation.[10-18] Approximate formulations of the transport equation have also appeared in several studies. Finite difference and finite element methods have been successfully implemented in solving the heat equation while numerous approaches have been applied to the two forms of the radiative equation of transfer. An additional entry to the numerical scene over the past 10 years involves the application of Green's functions.[19,20] Boundary integral methods have also made an impact in the investigation of nonlinear heat transfer studies.[21] It is interesting to note that numerous numerical methods have been proposed for solving the transient radiative/conductive heat transfer problem, but few investigations have considered alternative formulations that may lead to simplified and unified numerical treatments.

Of particular interest to the present work are the studies by Sutton,[15] Tsai and Lin,[16] and Lii and Özisik,[17] since these investigations presented numerical results for the identical problem posed here. Sutton[15] developed a numerical method that blends an explicit finite difference method for transient conduction with a Galerkin approach for the radiative component. Tsai and Lin[16] used a finite difference/nodal method for solving the transient, combined-mode problem. Tsai and Lin[16] also presented a variety of tabulated results that can be used for comparison purposes. The numerical approach offered here significantly differs from these other works since the basic formulation is altered prior to the implementation of a numerical method. In this way, a unified, simplified, and consistent computational method can be proposed.

This article offers a new and unified formulation for mixed-mode, transient radiative and conductive transport in a participating medium where a common computational thread is interwoven into the formulation of the entire system of equations. Generalizations can easily be inferred by the reader through the systematic formulation and discussion offered here. This article is presented in six major sections that cover the development of the concept to the presentation of some initial findings. Section II presents the mathematical formulation of a classical situation and then presents a new formulation based on the introduction of cumulative variables.[22,23] Section III proposes a simple numerical method for solving the system of coupled, transient, integral, and differential equations. Section IV offers some preliminary calculations and comparisons with previously reported works. Finally, Sec. V offers some conclusions and recommendations for future considerations.

II. Mathematical Formulation

In the present context, consider the transient one-dimensional heat equation in the presence of both conductive and volumetric radiative effects in a plane-parallel, isotropically scattering, gray medium[11]

$$\frac{1}{\alpha^2}\frac{\partial^2\theta}{\partial\eta^2}(\eta,\xi) - \frac{1}{N_{cr}}(1-\omega)[\theta^4(\eta,\xi) - G(\eta,\xi)] = \frac{\partial\theta}{\partial\xi}(\eta,\xi), \qquad \eta\varepsilon(-1,1), \qquad \xi > 0 \tag{1a}$$

subject to the auxiliary conditions

$$\theta(-1,\xi) = 1 \tag{1b}$$

$$\theta(1,\xi) = \theta_2, \qquad \xi > 0 \tag{1c}$$

$$\theta(\eta,0) = \theta_i, \qquad \eta\varepsilon[-1,1] \tag{1d}$$

The isothermal boundary conditions are presented in Eqs. (1b) and (1c), while the initial condition is given in Eq. (1d). The spatial domain is presented in the region $\eta\varepsilon[-1, 1]$ in anticipation of the choice of orthogonal functions to be introduced in the next section.

The conventional integral form of the radiative equation of transfer in the presence of black surfaces can be written as

$$G(\eta,\xi) = \frac{1}{2}\left\{E_2[\alpha(1+\eta)] + \theta_2^4 E_2[\alpha(1-\eta)] + \alpha\int_{\eta_0=-1}^{1}[(1-\omega)\theta^4(\eta_0,\xi) + \omega G(\eta_0,\xi)]E_1(\alpha|\eta-\eta_0|)\,d\eta_0\right\} \tag{2}$$

Here, $E_n(z)$ represents the nth exponential integral function[24] where $E_1(z)$ contains a well-known logarithmic (weak) singularity as $z \to 0$.

Before proceeding further, two dimensionless flux relations are presented. The dimensionless net radiative heat flux can be expressed as

$$Q^r(\eta,\xi) = \frac{1}{2}\left\{E_3[\alpha(1+\eta)] - \theta_2^4 E_3[\alpha(1-\eta)] + \alpha\int_{\eta_0=-1}^{\eta}[(1-\omega)\theta^4(\eta_0,\xi) + \omega G(\eta_0,\xi)] \times E_2[\alpha(\eta-\eta_0)]\,d\eta_0 - \alpha\int_{\eta_0=\eta}^{1}[(1-\omega)\theta^4(\eta_0,\xi) + \omega G(\eta_0,\xi)]E_2[\alpha(\eta_0-\eta)]\,d\eta_0\right\} \tag{3}$$

while the dimensionless net heat flux can be written as

$$Q(\eta,\xi) = -\frac{1}{\alpha}\frac{\partial\theta}{\partial\eta}(\eta,\xi) + \frac{1}{N_{cr}}Q^r(\eta,\xi), \qquad \eta\varepsilon[-1,1], \qquad \xi \geq 0 \tag{4}$$

The cumulative variable approach[23] begins by a partial decomposition of the differential operators displayed in Eq. (1). That is, the approach begins by integrating the temporal variable ξ in both the heat equation displayed in Eq. (1) to get

$$\theta(\eta,\xi) = \theta_i + \frac{1}{\alpha^2}\frac{\partial^2\Psi_1}{\partial\eta^2}(\eta,\xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta,\xi) - \Psi_3(\eta,\xi)], \qquad \eta(-1,1), \qquad \xi \geq 0 \tag{5a}$$

and in the integral form of the radiative equation of transfer, as displayed in Eq. (2), to arrive at

$$\Psi_3(\eta,\xi) = h(\eta,\xi) + \frac{\alpha}{2}\int_{\eta_0=-1}^{1}[(1-\omega)\Psi_2(\eta_0,\xi) + \omega\Psi_3(\eta_0,\xi)]E_1(\alpha|\eta-\eta_0|)\,d\eta_0, \qquad \eta\varepsilon[-1,1], \qquad \xi \geq 0 \tag{5b}$$

with

$$h(\eta,\xi) = (\xi/2)\{E_2[\alpha(1+\eta)] + \theta_2^4 E_2[\alpha(1-\eta)]\} \tag{5c}$$

and where the three new cumulative variables $\{\Psi_k(\eta, \xi)\}_{k=1}^3$ are defined as

$$\Psi_1(\eta, \xi) \equiv \int_{\xi_0=0}^{\xi} \theta(\eta, \xi_0)\, d\xi_0 \tag{6a}$$

$$\Psi_2(\eta, \xi) \equiv \int_{\xi_0=0}^{\xi} \theta^4(\eta, \xi_0)\, d\xi_0 \tag{6b}$$

$$\Psi_3(\eta, \xi) \equiv \int_{\xi_0=0}^{\xi} G(\eta, \xi_0)\, d\xi_0 \tag{6c}$$

This formulation is clearly in contrast with other past investigations,[1-18] and even those using a Green's function or boundary element approach,[21] since only the temporal portion of the linear operator is inverted in the present context. This reformulation blends the concept offered by Frankel and Choudhury[22] when investigating the integrodifferential problem of Volterra,[26] and the notion offered by Kumar and Sloan[25] when investigating Hammerstein integral equations. Next, Eqs. (6a–6c) are differentiated with respect to the temporal variable ξ to get

$$\frac{\partial \Psi_1}{\partial \xi}(\eta, \xi) = \theta(\eta, \xi) \tag{7a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(\eta, \xi) = \theta^4(\eta, \xi) \tag{7b}$$

$$\frac{\partial \Psi_3}{\partial \xi}(\eta, \xi) = G(\eta, \xi) \tag{7c}$$

respectively. Upon substituting Eq. (5a) into Eqs. (7a) and (7b), we obtain

$$\frac{\partial \Psi_1}{\partial \xi}(\eta, \xi) = \theta_i + \frac{1}{\alpha^2}\frac{\partial^2 \Psi_1}{\partial \eta^2}(\eta, \xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta, \xi) - \Psi_3(\eta, \xi)], \quad \eta(-1, 1), \quad \xi > 0 \tag{8a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(\eta, \xi) = \left\{\theta_i + \frac{1}{\alpha^2}\frac{\partial^2 \Psi_1}{\partial \eta^2}(\eta, \xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta, \xi) - \Psi_3(\eta, \xi)]\right\}^4, \quad \eta(-1, 1), \quad \xi > 0 \tag{8b}$$

Equations (5b), (8a), and (8b) represent the new system of equations from which a unified numerical method can be implemented. It is easy to see through viewing Eqs. (6a–6c) that the initial conditions for the cumulative variables $\{\Psi_k(\eta, \xi)\}_{k=1}^3$ are

$$\Psi_k(\eta, 0) = 0, \qquad k = 1, 2, 3, \qquad \eta\varepsilon[-1, 1] \tag{9}$$

The known temperature boundary conditions can be recast into the cumulative variable formulation through Eqs. (7a) and (7b)

$$\frac{\partial \Psi_1}{\partial \xi}(-1, \xi) = 1 \tag{10a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(-1, \xi) = 1 \tag{10b}$$

$$\frac{\partial \Psi_1}{\partial \xi}(1, \xi) = \theta_2 \tag{10c}$$

$$\frac{\partial \Psi_2}{\partial \xi}(1, \xi) = \theta_2^4, \qquad \xi > 0 \tag{10d}$$

The mathematical formulation in the cumulative variables is now well-posed and ready for further analysis.

Some important observations can be made contrasting the conventional formulation as indicated by Eqs. (1) and (2) with the new cumulative variable formulation as presented in Eqs. (5b), (8a), and (8b). The most apparent differences between the conventional formulation and the present notion lies in the observation that 1) the present system of equations is unified in structure and can be resolved using a single and consistent numerical method, 2) the nonlinearities are rearranged into positions conducive to orthogonal collocation, and 3) three dependent variables are present in the new formulation rather than the usual two dependent variables associated with the conventional formulation.

III. Computational Methodology

In this section, an orthogonal collocation method is presented for finding an approximate solution to $\Psi_k(\eta, t)$, $k = 1, 2, 3$ as shown in Eqs. (5b), (8a), and (8b) and subject to the initial conditions shown in Eq. (9) and boundary conditions displayed in Eqs. (10a–10d). The physical variables can be reconstructed once the cumulative variables are resolved satisfactorily.

Let the unknown functions $\Psi_k(\eta, t)$, $k = 1, 2, 3$ be formally represented by the series expansions

$$\Psi_1(\eta, \xi) = \sum_{m=0}^{\infty} a_m(\xi) T_m(\eta) \tag{11a}$$

$$\Psi_2(\eta, \xi) = \sum_{m=0}^{\infty} b_m(\xi) T_m(\eta) \tag{11b}$$

$$\Psi_3(\eta, \xi) = \sum_{m=0}^{\infty} c_m(\xi) T_m(\eta), \quad \eta\varepsilon[-1, 1], \quad \xi \geq 0 \tag{11c}$$

where the basis functions $\{T_m(\eta)\}_{m=0}^{\infty}$ are chosen as the Chebyshev polynomials of the first kind,[13,17] and are expressible as

$$T_m(\eta) = \cos[m(\cos^{-1}\eta)], \qquad m = 0, 1, \ldots, N \tag{12}$$

Other forms for the expansions of $\{\Psi(\eta, \xi)\}_{k=1}^3$ are also possible. These forms may include terms that account for the boundary conditions. Chebyshev polynomials have numerous exploitable features[27,28] and have successfully been used in the studies involving fluid mechanics,[29] solid mechanics,[30] conduction,[31] and radiative transport.[23,32] The unknown time-varying expansion coefficients requiring resolution are denoted as $\{a_m(\xi), b_m(\xi), c_m(\xi)\}_{m=0}^{\infty}$. In practice, we must truncate this series representation at a finite number of terms, say at order N. Thus, we denote the Nth-order approximation to $\Psi_k(\eta, \xi)$ as $\Psi_k^N(\eta, \xi)$, $k = 1, 2, 3$, viz.,

$$\Psi_1(\eta, \xi) \approx \Psi_1^N(\eta, \xi) = \sum_{m=0}^{N} a_m^N(\xi) T_m(\eta) \tag{13a}$$

$$\Psi_2(\eta, \xi) \approx \Psi_2^N(\eta, \xi) = \sum_{m=0}^{N} b_m^N(\xi) T_m(\eta) \tag{13b}$$

$$\Psi_3(\eta, \xi) \approx \Psi_3^N(\eta, \xi) = \sum_{m=0}^{N} c_m^N(\xi) T_m(\eta), \quad \eta\varepsilon[-1, 1], \quad \xi \geq 0 \tag{13c}$$

where $a_m^N(\xi)$, $b_m^N(\xi)$, $c_m^N(\xi)$ represent approximations to $a_m(\xi)$, $b_m(\xi)$, $c_m(\xi)$, respectively, for each fixed m in the finite set.

Upon substituting the series representations shown in Eqs.

(13a–13c) for $\{\Psi_k^N(\eta, \xi)\}_{k=1}^{3}$ into Eqs. (5b), (8a), and (8b), we arrive at

$$R_1^N(\eta, \xi) + \sum_{m=0}^{N} \frac{da_m^N}{d\xi}(\xi)T_m(\eta) = \theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta) - \frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta) \tag{14a}$$

$$R_2^N(\eta, \xi) + \sum_{m=0}^{N} \frac{db_m^N}{d\xi}(\xi)T_m(\eta) = \left\{\theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta) - \frac{(1-\omega)}{N_{cr}} \times \sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta)\right\}^4; \quad \eta\varepsilon(-1, 1), \quad \xi > 0 \tag{14b}$$

$$R_3^N(\eta, \xi) + \sum_{m=0}^{N} c_m^N(\xi)T_m(\eta) = h(\eta, \xi) + \frac{\alpha(1-\omega)}{2}\sum_{m=0}^{N} b_m^N(\xi)A_m^\alpha(\eta) + \frac{\alpha\omega}{2}\sum_{m=0}^{N} c_m^N(\xi)A_m^\alpha(\eta) \quad \eta\varepsilon[-1, 1], \quad \xi \ge 0 \tag{14c}$$

respectively, and where $A_m^\alpha(\eta)$, $m = 0, 1, \ldots, N$ is defined as

$$A_m^\alpha(\eta) \equiv \int_{\eta_0=-1}^{1} T_m(\eta_0)E_1(\alpha|\eta - \eta_0|)\, d\eta_0 \tag{14d}$$

which can be analytically integrated to yield

$$A_m^\alpha(\eta) = -\sum_{k=0}^{m}\frac{1}{\alpha^{k+1}}\{(-1)^k T_m^{(k)}(-1)E_{k+2}[\alpha(1+\eta)] + T_m^{(k)}(1)E_{k+2}[\alpha(1-\eta)]\} + 2\sum_{j=0}^{[m/2]}\frac{1}{\alpha^{2j+1}} \times T_m^{(2j)}(\eta)E_{2j+2}(0), \quad m = 0, 1, \ldots, N \tag{14e}$$

Here, we denote the kth derivative of the mth Chebyshev polynomial of the first kind as $T_m^{(k)}(\eta)$, and where $[m/2]$ is interpreted as the integer result of $(m/2)$. Here, $R_k^N(\eta, \xi)$, $k = 1, 2, 3$ represent the local and instantaneous residual functions as required in order to maintain the equality displayed in Eqs. (14a–14c).

Using the finite series representations for $\Psi_k^N(\eta, \xi)$, $k = 1, 2$, in conjunction with the specified boundary conditions shown by Eqs. (10a–10d), we obtain

$$R_1^N(-1, \xi) + \sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(-1) = 1 \tag{15a}$$

$$R_1^N(1, \xi) + \sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(1) = \theta_2 \tag{15b}$$

$$R_2^N(-1, \xi) + \sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(-1) = 1 \tag{15c}$$

$$R_2^N(1, \xi) + \sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(1) = \theta_2^4, \quad \xi > 0 \tag{15d}$$

while the appropriate initial conditions shown in Eq. (9) for $k = 1, 2$ become

$$R_1(\eta, 0) + \sum_{m=0}^{N} a_m^N(0)T_m(\eta) = 0 \tag{15e}$$

$$R_2(\eta, 0) + \sum_{m=0}^{N} b_m^N(0)T_m(\eta) = 0, \quad \eta\varepsilon[-1, 1] \tag{15f}$$

Unless the exact solution to $\Psi_k(\eta, t)$, $k = 1, 2, 3$ at any instant in time, $\xi \ge 0$, is a linear combination of $\{T_m(\eta)\}_{m=0}^{N}$, we cannot obtain $\{a_m^N(\xi), b_m^N(\xi), c_m^N(\xi)\}_{m=0}^{N}$, which makes $R_k^N(\eta, \xi)$ vanish for $k = 1, 2, 3$, $\xi \ge 0$ and $\eta\varepsilon[-1, 1]$. However, we can obtain suitable time varying expansion coefficients by making the residuals indicated in Eqs. (14a–14c) and (15a–15f) small in some sense.

For the collocation method, the orthogonality relation[23] becomes

$$\langle R_k^N(\eta, \xi), \Omega_j(\eta)\rangle_{w_j} = 0, \quad k = 1, 2, 3, \quad \xi \ge 0 \tag{16}$$

where $w_j = 1$, $\Omega_j(\eta) = \delta(\eta - \eta_j)$, $j = 0, 1, \ldots, N$. Here, the Dirac delta function is denoted by δ, while the $N + 1$ collocation points are indicated by η_j, $j = 0, 1, \ldots, N$ and are defined by the closed rule[28]

$$\eta_j = \cos(\pi j/N), \quad j = 0, 1, \ldots, N \tag{17}$$

By choosing this set of $N + 1$ collocation points, we ensure that $R_k^N(\pm 1, \xi) = 0$, $k = 1, 2, 3$ for $\xi > 0$ in Eqs. (14a–14c). Note that from Eq. (17), one interprets that $\eta_0 = 1$ and $\eta_N = -1$. Error and convergence analyses[33] have been performed illustrating the merit of this choice of basis functions and collocation points in the study of the radiative equation of transfer in an isotropically scattering medium.

Applying the orthogonality condition displayed by Eq. (16) to Eq. (14a) for $j = 1, \ldots, N - 1$ produces

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_j) = \theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta_j) - \frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta_j) \quad j = 1, 2, \ldots, N-1 \tag{18a}$$

along with imposing the boundary constraints

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_0) = \theta_2 \tag{18b}$$

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_N) = 1, \quad \xi > 0 \tag{18c}$$

Upon similar application of Eq. (16) on Eq. (14b), we arrive at

$$\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(\eta_j) = \left\{\theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta_j) - \frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta_j)\right\}^4 \quad j = 1, 2, \ldots, N-1 \tag{18d}$$

along with imposing the boundary constraints

$$\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(\eta_0) = \theta_2^4 \tag{18e}$$

$$\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(\eta_N) = 1, \quad \xi > 0 \tag{18f}$$

while Eq. (14c) reduces to

$$\sum_{m=0}^{N} c_m^N(\xi)T_m(\eta_j) = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2}\sum_{m=0}^{N} b_m^N(\xi)A_m^\alpha(\eta_j) + \frac{\alpha\omega}{2}\sum_{m=0}^{N} c_m^N(\xi)A_m^\alpha(\eta_j), \quad \xi \ge 0, \quad j = 0, 1, \ldots, N \tag{18g}$$

Under the orthogonality relation defined by Eq. (16), the initial conditions displayed in Eqs. (15e) and (15f) reduce to

$$a_m^N(0) = b_m^N(0) = 0, \quad m = 0, 1, \ldots, N, \quad \eta \epsilon [-1, 1] \tag{18h}$$

Equation (18g) can be alternately expressed as

$$\sum_{m=0}^{N} c_m^N(\xi)[T_m(\eta_j) - \frac{\alpha\omega}{2} A_m^\alpha(\eta_j)] = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2} \sum_{m=0}^{N} b_m^N(\xi) A_m^\alpha(\eta_j)$$
$$\xi \geq 0, \quad j = 0, 1, \ldots, N \tag{19}$$

which naturally leads to the matrix form $A\bar{c}(\xi) = \bar{f}(\xi)$ where the vector $\bar{c}(\xi)$ contains the unknown time varying coefficients, i.e., $\bar{c}(\xi) = [c_0^N(\xi), c_1^N(\xi), \ldots, c_N^N(\xi)]^T$. Clearly, we can express $\bar{c}(\xi)$ as

$$\bar{c}(\xi) = A^{-1}\bar{f}(\xi), \quad \xi \geq 0 \tag{20a}$$

when $|A| \neq 0$, and where A is given by

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{bmatrix} \tag{20b}$$

while $\bar{f}(\xi) = [f_0^N(\xi), f_1^N(\xi), \ldots, f_N^N(\xi)]^T$. The components of the coefficient matrix are thus

$$a_{jm} = T_m(\eta_j) - (\alpha\omega/2)A_m^\alpha(\eta_j), \quad j = 0, 1, \ldots, N$$
$$m = 0, 1, \ldots, N \tag{20c}$$

while the unknown time varying components of $\bar{f}(\xi)$ are given as

$$f_j^N(\xi) = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2} \sum_{m=0}^{N} b_m^N(\xi) A_m^\alpha(\eta_j)$$
$$j = 0, 1, \ldots, N \tag{20d}$$

This is quite useful since we can analytically eliminate $\bar{c}(\xi) = [c_0^N(\xi), c_1^N(\xi), \ldots, c_N^N(\xi)]^T$ from Eqs. (18a) and (18d), which alleviates the need to determine these coefficients explicitly. Thus, we are left with two matrix differential equations requiring resolution for the time varying expansion coefficients $a_m^N(\xi)$, $b_m^N(\xi)$, $m = 0, 1, \ldots, N$ instead of two matrix differential equations and an algebraic system.

Likewise, Eqs. (18a–18f) can be expressed, using compact matrix notation, as

$$B \frac{d\bar{a}(\xi)}{d\xi} = \bar{g}(\xi) \tag{21a}$$

$$B \frac{d\bar{b}(\xi)}{d\xi} = \bar{h}(\xi), \quad \xi > 0 \tag{21b}$$

where the common and known $(N + 1) \times (N + 1)$ coefficient matrix B is

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0N} \\ b_{10} & b_{11} & \cdots & b_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N0} & b_{N1} & \cdots & b_{NN} \end{bmatrix} \tag{22a}$$

where each entry is given by

$$b_{jm} = T_m(\eta_j), \quad j = 0, 1, \ldots, N, \quad m = 0, 1, \ldots, N \tag{22b}$$

while the vectors $\bar{g}(\xi) = [g_0^N(\xi), g_1^N(\xi), \ldots, g_N^N(\xi)]^T$ and $\bar{h}(\xi) = [h_0^N(\xi), h_1^N(\xi), \ldots, h_N^N(\xi)]^T$ have components

$$g_0^N(\xi) = \theta_2 \tag{22c}$$

$$g_j^N(\xi) = \theta_i + \frac{1}{\alpha^2} \sum_{m=0}^{N} a_m^N(\xi) T_m''(\eta_j) - \frac{(1-\omega)}{N_{cr}} \sum_{m=0}^{N} [b_m^N(\xi) - c_m^N(\xi)] T_m(\eta_j), \quad j = 1, 2, \ldots, N-1 \tag{22d}$$

$$g_N^N(\xi) = 1 \tag{22e}$$

while

$$h_j^N(\xi) = [g_j^N(\xi)]^4, \quad j = 0, 1, \ldots, N \tag{22f}$$

The vectors defined as $\bar{a}(\xi)$ and $\bar{b}(\xi)$ are expressible as $\bar{a}(\xi) = [a_0^N(\xi), a_1^N(\xi), \ldots, a_N^N(\xi)]^T$ and $\bar{b}(\xi) = [b_0^N(\xi), b_1^N(\xi), \ldots, b_N^N(\xi)]^T$. Inverting the known coefficient matrix B in Eqs. (21a) and (21b) produces the system of nonlinear initial value problems requiring numerical approximation. Once the time varying coefficients $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$ for $\xi \geq 0$ are determined then the approximation of the necessary field variables $\theta_N(\eta, \xi)$ and $Q_N^r(\eta, \xi)$ can be obtained.

Three physical variables can now be recovered without major effort. Indeed, the majority of the computational effort lies in calculating the unknown expansion coefficients, $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$. The physical variables are constructed during a postprocessing procedure and require minimal computational effort. It is advantageous to reconstruct the approximate solution for the dimensionless temperature variable $\theta_N(\eta, \xi)$ using Eq. (7a), namely

$$\theta_N(\eta, \xi) = \frac{\partial \Psi_1^N}{\partial \xi}(\eta, \xi) = \sum_{m=0}^{N} \frac{da_m^N}{d\xi}(\xi) T_m(\eta) \tag{23}$$

The inversion formula shown by Eq. (5a) also will produce results identical to Eq. (23) at the interior collocation points. However, at the endpoints Eq. (5a) will not reproduce the imposed boundary conditions. This makes sense since the boundary conditions are manually imposed (overlayed) into Eqs. (21a) and (21b). The right side of Eq. (21a), which has been precalculated, can be used in determining $da_m^N(\xi)/d\xi$, $m = 0, 1, \ldots, N$ at the discrete times corresponding with the numerical integrator. Inverting the coefficient matrix B shown in Eq. (21a) allows us to obtain numerical values for $da_m^N(\xi)/d\xi$, $m = 0, 1, \ldots, N$. Note that B^{-1} has already been used and previously stored.

As indicated in Eq. (3), both $\theta^4(\eta, \xi)$ and $G(\eta, \xi)$ are required in establishing the local radiative heat flux $Q^r(\eta, \xi)$. Following a similar procedure as outlined previously for determining $\theta(\eta, \xi)$, one expresses the approximation of $\theta^4(\eta, \xi)$ as

$$\theta_N^4(\eta, \xi) = \frac{\partial \Psi_2^N}{\partial \xi}(\eta, \xi) = \sum_{m=0}^{N} \frac{db_m^N}{d\xi}(\xi) T_m(\eta) \tag{24}$$

An alternative approach is now offered for determining $G(\eta, \xi)$. This is broached since the approximation displayed by Eq. (13c) would require us to differentiate $\Psi_3^N(\eta, \xi)$ with respect to ξ. This is clearly undesirable since an additional approximation would be incurred in obtaining $dc_m^N(\xi)/d\xi$, $m = 0, 1, \ldots, N$. Let the approximation for $G(\eta, \xi)$ be written as $G_N(\eta, \xi)$ and expressed by

$$G_N(\eta, \xi) = \sum_{m=0}^{N} \gamma_m^N(\xi) T_m(\eta) \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (2), produces

$$R_4^N(\eta, \xi) + \sum_{m=0}^{N} \gamma_m^N(\xi) T_m(\eta) = \frac{1}{2}\Big\{E_2[\alpha(1+\eta)] + \theta_2^4 E_2[\alpha(1-\eta)] + \alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta) + \alpha\omega\sum_{m=0}^{N}\gamma_m^N(\xi)A_m^\alpha(\eta)\Big\} \tag{26a}$$

Clearly, $\gamma_m^N(\xi) = dc_m^N(\xi)/d\xi$, $m = 0, 1, \ldots, N$ and can be explicitly calculated using previously determined results. Applying the orthogonality relation shown in Eq. (16) on Eq. (26a) produces

$$\sum_{m=0}^{N}\gamma_m^N(\xi)\left[T_m(\eta_j) - \frac{\alpha\omega}{2}A_m^\alpha(\eta_j)\right] = \frac{1}{2}\Big\{E_2[\alpha(1+\eta_j)] + \theta_2^4 E_2[\alpha(1-\eta_j)] + \alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta_j)\Big\}$$

$$j = 0, 1, \ldots, N \tag{26b}$$

which can be written in the compact matrix form as

$$A\bar{\gamma}(\xi) = \bar{p}(\xi), \qquad \xi \geq 0 \tag{26c}$$

where $\bar{\gamma}(\xi)$ denotes the solution vector defined as $\bar{\gamma}(\xi) = [\gamma_0^N(\xi), \gamma_1^N(\xi), \ldots, \gamma_N^N(\xi)]^T$. The constant coefficient matrix A is identical to that previously expressed by Eq. (20b). At this point, the vector $\bar{p}(\xi) = [p_0^N(\xi), p_1^N(\xi), \ldots, p_N^N(\xi)]^T$ is completely known at a finite set of discrete times. Each component in the vector $\bar{p}(\xi)$ is given by

$$p_j^N(\xi) = \frac{1}{2}\left[E_2[\alpha(1+\eta_j)] + \theta_2^4 E_2[\alpha(1-\eta_j)] + \alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta_j)\right], \quad j = 0, 1, \ldots, N \tag{26d}$$

Clearly, $\bar{\gamma}(\xi)$ is easily recoverable through matrix inversion at the discrete times corresponding to the numerical integrator, namely

$$\bar{\gamma}(\xi) = A^{-1}\bar{p}(\xi) \tag{26e}$$

Using these variables, the local radiative heat flux, as shown by Eq. (3), can be written as

$$Q^r(\eta, \xi) \approx Q_N^r(\eta, \xi) = \frac{1}{2}\Big\{E_3[\alpha(1+\eta)] - \theta_2^4 E_3[\alpha(1-\eta)] + \alpha\sum_{m=0}^{N}\left[(1-\omega)\frac{db_m^N}{d\xi}(\xi) + \omega\gamma_m^N(\xi)\right]W_{1,m}^\alpha(\eta) - \alpha\sum_{m=0}^{N}\left[(1-\omega)\frac{db_m^N}{d\xi}(\xi) + \omega\gamma_m^N(\xi)\right]W_{2,m}^\alpha(\eta)\Big\}$$

$$\eta \varepsilon [-1, 1], \quad \xi \geq 0 \tag{27a}$$

where the functions $W_{1,m}^\alpha(\eta)$, $W_{2,m}^\alpha(\eta)$ for $m = 0, 1, \ldots, N$ are defined as

$$W_{1,m}^\alpha(\eta) \equiv \int_{\eta_0=-1}^{\eta} T_m(\eta_0)E_2[\alpha(\eta-\eta_0)]\,d\eta_0 \tag{27b}$$

$$W_{2,m}^\alpha(\eta) \equiv \int_{\eta_0=\eta}^{1} T_m(\eta_0)E_2[\alpha(\eta_0-\eta)]\,d\eta_0 \tag{27c}$$

which analytically integrate to

$$W_{1,m}^\alpha(\eta) = \sum_{i=0}^{m}\frac{(-1)^i}{\alpha^{i+1}}\{T_m^{(i)}(\eta)E_{i+3}(0) - T_m^{(i)}(-1)E_{i+3}[\alpha(1+\eta)]\} \tag{27d}$$

$$W_{2,m}^\alpha(\eta) = \sum_{i=0}^{m}\frac{1}{\alpha^{i+1}}\{T_m^{(i)}(1)E_{i+3}[\alpha(1-\eta)] - T_m^{(i)}(\eta)E_{i+3}(0)\}, \qquad m = 0, 1, \ldots, N \tag{27e}$$

At this point, all the ingredients are available to obtain numerical results for the temperature distribution and radiative, conductive, and total heat flux distributions.

IV. Preliminary Results

Preliminary findings are presented illustrating some computational features produced by this formulation. A program was written implementing the expressions and algorithm discussed in Sec. III using the symbolic software package Mathematica®[34] as implemented on a NeXT computer (NeXTstation Turbo). The computations required in the matrix manipulations were performed symbolically and left in analytic form. A fully explicit fifth-order, six-stage Runge–Kutta[35] was written in the Mathematica language. This initial value method merely serves to produce preliminary results. This method will be replaced by an implicit method in order to remove the stability constraints associated with an explicit method.

Some unique features of the new formulation are presented in order to elaborate on the computational attributes of the scheme. Figure 1 illustrates the nature of the time-varying expansion coefficients $a_m^N(\xi)$, $m = 0, 1, 2, 3, 4, 5$ when $N = 8$, $N_{cr} = 0.1$, $\alpha = 0.5$, $\omega = 0.5$, $\theta_i = \theta_2 = 0$. Accurate representations of these coefficients are critical in reconstructing the cumulative variable $\Psi_1^N(\eta, \xi)$. The well-behaved nature of these functions under the imposed constraints is clearly illuminated by this figure. This type of behavior lends itself to a good approximation. As $\xi \to \infty$, it appears that these coefficients grow no worse than linearly with respect to time. This observation has theoretical basis.[23] It should be noted that the fully explicit numerical integrator's stability constraint becomes evident as N increases, thus requiring smaller time steps to be used in order for the numerical method to remain stable. This minor inconvenience can be easily rectified by changing the initial value numerical method.[35]

The development of rigorous error estimates for this formulation is presently under consideration. As a preliminary indicator, comparison with other published works is offered. Frankel has developed rigorous error estimates[23,33] and convergence rates[33] using this set of basis functions when investigating an integral form of the transport equation for steady-

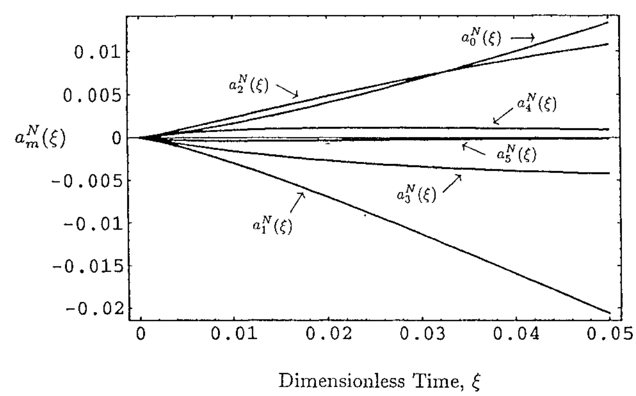

Fig. 1 Time-varying expansion coefficients $a_m^N(\xi)$, $m = 0, 1, 2, 3, 4, 5$, when $N = 8$, $N_{cr} = 0.1$, $\omega = 0.5$, $\alpha = 0.5$, and $\theta_i = \theta_2 = 0$.

Table 1 Comparison of temperature results at three spatial locations when $\xi = 0.05$ and where $N_{cr} = 0.1$, $\omega = 0.5$, $\alpha = 0.5$, and $\theta_l = \theta_2 = 0$

Investigators[16]	Dimensionless temperatures $\eta = -0.5$	$\eta = 0$	$\eta = 0.5$
Lii and Özisik	0.4617	0.1474	0.0277
Sutton	0.4888	0.1778	0.0591
Barker and Sutton	0.4893	0.1775	0.0588
Tsai and Lin	0.4889	0.1773	0.0588
Present study: $\theta_N(\eta, 0.05)$			
$N = 4$	0.4996	0.1797	0.0504
$N = 6$	0.4888	0.1777	0.0584
$N = 8$	0.4893	0.1773	0.0587

Table 2 Comparison of radiative heat flux results at three spatial locations when $\xi = 0.05$ and where $N_{cr} = 0.1$, $\omega = 0.5$, $\alpha = 0.5$, and $\theta_l = \theta_2 = 0$

Investigators[16]	Dimensionless radiative heat fluxes $\eta = -1$	$\eta = 0$	$\eta = 1$
Lii and Özisik	1.6436	1.2529	0.9746
Sutton	1.9304	1.3305	0.8332
Barker and Sutton	1.9300	1.3314	0.8335
Tsai and Lin	1.9328	1.3292	0.8321
Present study: $Q_N^r(\eta, 0.05)/N_{cr}$			
$N = 4$	1.9355	1.3025	0.8339
$N = 6$	1.9348	1.3284	0.8317
$N = 8$	1.9342	1.3289	0.8319

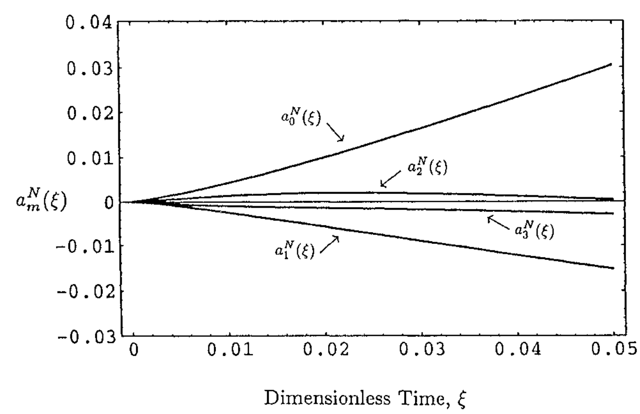

Fig. 2 Time-varying expansion coefficients $a_m^N(\xi)$, $m = 0, 1, 2, 3$, when $N = 8$, $N_{cr} = 0.01$, $\omega = 0.2$, $\alpha = 0.5$, and $\theta_l = \theta_2 = 0.25$.

state radiative transport in a plane-parallel geometry. In transient studies, it is possible to develop estimates based on truncation errors associated with Runge–Kutta methods[35] in the temporal variable. Unfortunately, in complicated physical problems it is often difficult to obtain realistic error estimates. Thus, without rigorous error estimates, only qualitative assessments are available.

As an indicator of numerical accuracy, Tables 1 and 2 are presented comparing several reported works to the present analysis. The set of system parameters chosen corresponds to an example with a wealth of tabular results.[16] Comparison of the present method to other investigations is quite favorable, as shown in Tables 1 and 2, for the case where $N = 8$, $N_{cr} = 0.1$, $\alpha = 0.5$, $\omega = 0.5$, $\theta_l = \theta_2 = 0$, and $\xi = 0.05$. Table 1 presents temperature values at three interior locations. Agreement is apparent when compared with the reported results of Sutton, Barker and Sutton, and Tsai and Lin, as taken from Ref. 16. Note that the underlined entries in these tables indicate locations that coincide with the collocation points. Correspondingly good results for the net radiative heat flux are displayed in Table 2. The present conclusion is that the proposed methodology produces comparable numerical results to that reported by other studies. Other cases have been considered and compared to tabulated results when available in order to validate the proposed methodology.

The second example considers the situation where $N_{cr} = 0.01$, $\omega = 0.2$, $\alpha = 0.5$, $\theta_l = \theta_2 = 0.25$, which was previously reported by Sutton.[15] Figure 2 displays the time-varying expansion coefficients $a_m^N(\xi)$, $m = 0, 1, 2, 3$, when $N = 8$. Similar qualitative features to that of the previous case are indicated in this figure. Figure 3 presents the constructed temperature distribution $\theta_N(\eta, \xi)$ at four distinct times $\xi = 0.005, 0.01, 0.025, 0.05$. Meanwhile, Fig. 4 shows the corresponding radiative heat flux distribution $Q_N^r(\eta, \xi)$ at the identical times. This figure produces identical graphical results to that illustrated by Sutton.[15] Letting $N = 6$ in the present formulation produces graphically identical results to $N = 8$. Finally, the proposed method works exceptionally well for the cases of $\omega = 0, 1$.

Some additional remarks are offered concerning the development of the spatial temperature and flux distributions. The numerical solution at the collocation points $\{\eta_k\}_{k=0}^N$ will typically be more accurate than at other spatial locations,[23,33] owing in part to the oscillatory behavior of the residual function that is well documented.[23,33] Numerical values for the

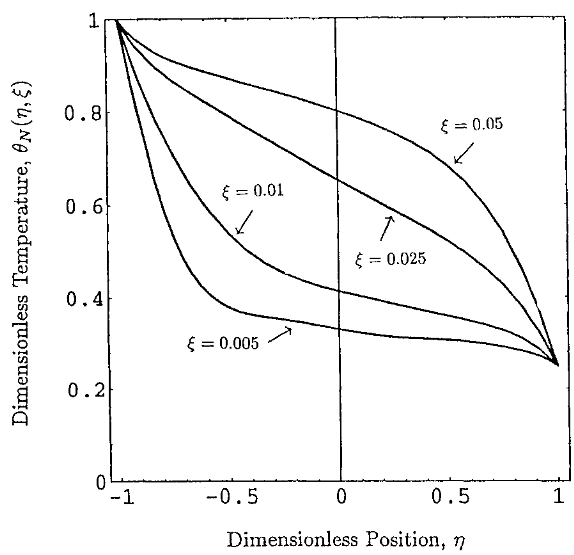

Fig. 3 Temperature distributions at the four indicated times ξ corresponding to the physical parameters indicated in Fig. 2.

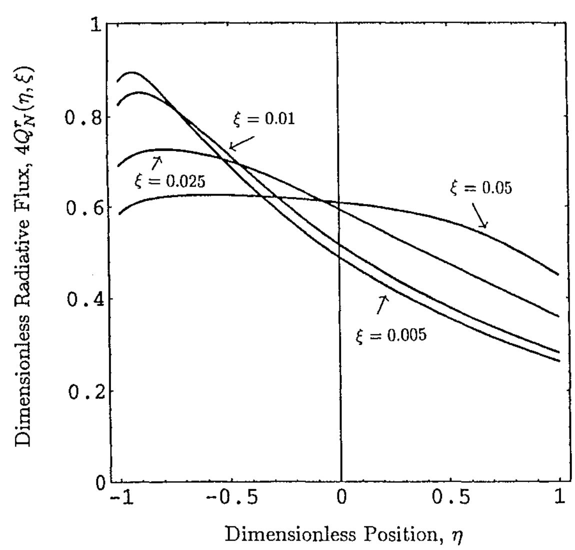

Fig. 4 Radiative heat flux distributions at the four indicated times ξ corresponding to the physical parameters indicated in Fig. 2.

temperatures and radiative heat fluxes at noncollocation positions were obtained through Eqs. (23) and (27a), respectively. An alternative procedure that has yet to be explored involves the use of a least-squares method for developing an appropriate polynomial approximation based on minimizing the deviations from the proposed curve fit to the predefined values of the function at the discrete collocation points. An orthogonal basis would be proposed since this choice could alleviate the well-known potential for ill-conditioning associated with the coefficient matrix in the linear system (Ref. 36, p. 641). This method would certainly produce fast numerical results when desiring numerical values for the radiative heat flux. Also, careful monitoring of the condition number and the determination of the optimal order of the curve fit based on the variance could be incorporated into a simulation package.

V. Conclusions

The objective of this article was to present a unified formulation that renders a systematic and unified numerical treatment to transient combined conductive/radiative transport in a participating medium. The numerical procedure postulated and then demonstrated uses a common computational theme for solving the heat equation and equation of radiative transport. Generalization of the present concept to include anisotropic scattering[37] and multidimensional situations will be addressed in later investigations. As indicated in Ref. 37, the number of simultaneous integral equations requiring resolution increases with the degree of anisotropy. However, little additional complication is introduced to the numerical procedure since these equations retain the mathematical structure of Eqs. (5b) and (6c). Simulation run times will increase due to solving for additional expansion coefficients as required in the approximation process. This is presently under investigation along with extending the procedure to multidimensional problems. Additionally, the incorporation of an implicit solver offers a major stability advantage over an explicit scheme and thus allows for substantially larger time steps to be used in the algorithm. It should be remarked that an iterative method must be introduced at each time step for purposes of resolving the expansion coefficients. Overall, a CPU savings is anticipated. It is also clear from viewing the physical behavior of the expansion coefficients reported in the present work that low-order block-by-block methods[38] may be appropriate. In effect, fast and accurate results may be available with this approach. This will also be reported on in an upcoming Note. Finally, it may be remarked that uniform approximations are often more desirable than piecewise approximations, which have some degree of regularity at the connection points.

Acknowledgments

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